Anomalous Hall effect in IV-VI magnetic semiconductors

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Topological contribution to the anomalous Hall effect in IV-VI narrow-gap semiconductors with a nonzero spontaneous magnetization due to magnetic impurities is considered theoretically. The off-diagonal conductivity is calculated in the relativistic model of the IV-VI semiconductors. Spin-orbit interaction in these compounds is strong and cannot be treated perturbatively. Therefore, it is included in the Hamiltonian of a clean (defect-free) system. Geometrical interpretation of the topological contribution to the Hall conductivity is also briefly discussed.

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I. INTRODUCTION

There is great interest currently in the anomalous Hall effect[1] (AHE) in magnetic macroscopic systems and low-dimensional structures.[2] This interest is partly stimulated by the different physics underlying the effect as well as by the application possibilities—e.g., in magnetization measurements of thin magnetic films.[3] From the theoretical point of view, the standard approach to the AHE is based on the known models of side jump[4] and skew scattering[5] on impurities due to spin-orbit interaction. It has been shown recently in the relativistic Dirac model[2,6] that both mechanisms can be described in a unified way within a perturbative approximation with respect to the relativistic terms.

There was an extensive debate in recent literature on the physical origin of AHE, and several mechanisms contributing to AHE and not associated with scattering from impurities have been proposed.[7–10] One of them, the so-called *intrinsic mechanism* of AHE,[8,9,11,12] attracted much attention in recent years and triggered hot discussion. The key point of the intrinsic mechanism is related to the contribution from electronic states with energies below the Fermi level E_F. This contribution does not vanish in the case of nontrivial topology of the electron energy bands and can be much larger than the contribution due to conventional side jump or skew scattering on impurities. It turned out, however, that scattering from impurities can be crucial for the intrinsic mechanism, suppressing it considerably even at low density of impurities.[13–15]

An important point which has been clarified in the discussion[13,14] concerns separation of the off-diagonal conductivity σ_{xy} into two parts, σ_{xy}^{I} and $\sigma_{xy}^{\mathrm{II}}$, where σ_{xy}^{I} can be associated with the classical Drude conductivity, whereas $\sigma_{xy}^{\mathrm{II}}$ contains some purely nonclassical effects.[16] It should be noted that $\sigma_{xy}^{\mathrm{II}}$ includes contribution from states below the Fermi energy as well as at the Fermi surface itself and *does not depend* on impurities. In contrast, the part σ_{xy}^{I} is related only to the Fermi surface and depends strongly on the presence of impurities, so that the clean (ballistic) limit of σ_{xy}^{I} does not coincide with the limit of small impurity density.[13,14]

The intrinsic mechanism of AHE has been studied thoroughly for the model of two-dimensional (2D) electron gas with Rashba spin-orbit interaction.[13,15,17–19] The relativistic 2D Weyl model with the linear electron energy spectrum near the Dirac point also has been considered.[14]

In this paper, in turn, we focus on the three-dimensional (3D) Dirac model of energy spectrum, which can be used for a large class of narrow-gap IV-VI semiconductors.[20] Typical examples are the compounds PbTe, PbSe, SnTe, GeTe, as well as their alloys. The relativistic terms in these semiconductors are not small and cannot be treated perturbatively. These terms determine both the nonparabolicity of the energy spectrum and the spin-orbit interaction. When doped with magnetic ions (e.g., with Mn), such semiconductors may become spontaneously magnetized owing to the indirect exchange interaction between magnetic moments of the impurities.[21]

Recently, the IV-VI magnetic semiconductors attract much attention mainly from the experimental point of view.[22–24] The anomalous Hall conductivity has been measured for instance in PbSnMnTe, SnMnTe, and also in SnMnEuTe mixed crystals containing Mn ions up to 16 at. %. Any analysis of the experimental results should take into account peculiarities of energy spectrum of the IV-VI semiconductors. However, no theoretical and/or numerical considerations of the AHE in IV-VI semiconductors have been presented so far. Therefore, in this work we calculate the off-diagonal conductivity in IV-VI magnetic semiconductors using the Dirac model[20] of the relevant energy spectrum. More specifically, we limit considerations to the intrinsic (topological) contribution, which seems to play an important role in these systems. For simplicity, the calculations are restricted to the zero-temperature limit.

The paper is organized as follows. In Sec. II we describe the model and present calculations of the Hall conductivity at the Fermi level. This follows the ideas of splitting the Hall conductivity into two terms; one is due to electrons at the Fermi level and the second is from electrons below the Fermi level. In Sec. III, in turn, we calculate the full Hall conductivity of the system. By definition it also includes the contribution from electrons at the Fermi level. Geometrical inter-

pretation of the Hall conductivity is described in Sec. IV, while summary and final conclusions are in Sec. V. Appendixes A–C contain mathematical details of the calculations.

II. CONTRIBUTION TO THE ANOMALOUS HALL CONDUCTIVITY FROM THE FERMI SURFACE

We consider the Dirac model for electrons and holes in IV-VI magnetic semiconductors. Ferromagnetism in such systems is generated by indirect exchange interaction (via electrons or holes) between magnetic impurities (e.g., Mn) in otherwise nonmagnetic semiconductors. Assuming zero energy in the middle of the energy gap, the corresponding Hamiltonian can be written in the following matrix (4×4) form:

$$H=\begin{pmatrix}\Delta-g_cM\sigma_z & v_0\boldsymbol{\sigma}\cdot\mathbf{k}\\ v_0\boldsymbol{\sigma}\cdot\mathbf{k} & -\Delta-g_vM\sigma_z\end{pmatrix}, \tag{1}$$

where Δ is the half width of the energy gap, v_0 is the band interaction parameter, and M is the magnetization (measured in energy units), whereas g_c and g_v are dimensionless constants which describe magnetic splitting in the conduction and valence bands, respectively. The Pauli matrices $\sigma_x,\sigma_y,\sigma_z$ act in the spin space with the axis z assumed along the magnetization direction, and $\mathbf{k}$ is the wave vector.

To find the current-density operator $\hat{\mathbf{j}}$, we turn on the electromagnetic field $\mathbf{A}(t)$ by substituting $\mathbf{k}\to\mathbf{k}-e\mathbf{A}/\hbar c$ in Eq. (1) ($\hbar$ here is the Planck constant, c is the light velocity, and e is the electron charge), i.e., $H\to H_A$. The current-density operator takes then the form

$$\hat{\mathbf{j}}=-c\frac{\delta H_A}{\delta\mathbf{A}}=e\frac{v_0}{\hbar}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\equiv e\hat{\mathbf{v}}, \tag{2}$$

where $\hat{\mathbf{v}}$ is the electron velocity operator.

Using Eq. (2), the quantum-mechanical average current density can be written in the form

$$\mathbf{j}(\mathbf{r},t)=ie\frac{v_0}{\hbar}\mathrm{Tr}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}G(\mathbf{r},t;\mathbf{r},t+\delta), \tag{3}$$

where $G(\mathbf{r},t;\ \mathbf{r}',t')$ is the Green's function in the matrix form corresponding to the Hamiltonian H_A (including the electromagnetic field). After expanding the Green's function with respect to $\mathbf{A}(t)$ and Fourier transformation of the spatial and time variables, one arrives at the Kubo formula for the off-diagonal conductivity σ_{xy}^F at the Fermi level, which may be written in the following form:

$$\sigma_{xy}^F=\frac{e^2v_0^2}{2\pi\hbar}\mathrm{Tr}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\begin{pmatrix}0 & \sigma_x\\ \sigma_x & 0\end{pmatrix}G_{\mathbf{k}}^R(\varepsilon)\times\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix}G_{\mathbf{k}}^A(\varepsilon)|_{\varepsilon=\mu}, \tag{4}$$

where the energy ε is taken at the Fermi level, $\varepsilon=\mu$. In Eq. (4) $G_{\mathbf{k}}^R(\varepsilon)$ and $G_{\mathbf{k}}^A(\varepsilon)$ are the retarded and advanced Green's functions corresponding to Hamiltonian (1).

The nondiagonal conductivity vanishes in the nonmagnetic limit, $M\to0$. The leading term in the expansion of σ_{xy}^F with respect to M is the one linear in M. Therefore, the following considerations of this section are limited to σ_{xy}^F calculated in the linear approximation with respect to the magnetization M. Expanding the Green's functions G^R and G^A and taking the linear terms in M, one finds σ_{xy}^F in the form

$$\sigma_{xy}^F=-M\frac{e^2v_0^2}{2\pi\hbar}\mathrm{Tr}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\Bigg[\begin{pmatrix}0 & \sigma_x\\ \sigma_x & 0\end{pmatrix}G_{\mathbf{k}}^R(\varepsilon)\times\begin{pmatrix}g_c\sigma_z & 0\\ 0 & g_v\sigma_z\end{pmatrix}G_{\mathbf{k}}^R(\varepsilon)\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix}G_{\mathbf{k}}^A(\varepsilon)+\begin{pmatrix}0 & \sigma_x\\ \sigma_x & 0\end{pmatrix}G_{\mathbf{k}}^R(\varepsilon)\begin{pmatrix}0 & \sigma_y\\ \sigma_y & 0\end{pmatrix}G_{\mathbf{k}}^A(\varepsilon)\times\begin{pmatrix}g_c\sigma_z & 0\\ 0 & g_v\sigma_z\end{pmatrix}G_{\mathbf{k}}^A(\varepsilon)\Bigg]\Bigg|_{\varepsilon=\mu}. \tag{5}$$

The Green's functions $G^{R,A}$ in the above equation correspond to Hamiltonian (1) with $M=0$ and take the form

$$G_{\mathbf{k}}^{R,A}(\varepsilon)=\frac{1}{(\varepsilon\pm i\eta)^2-\epsilon_k^2}\begin{pmatrix}\varepsilon+\Delta & v_0\boldsymbol{\sigma}\cdot\mathbf{k}\\ v_0\boldsymbol{\sigma}\cdot\mathbf{k} & \varepsilon-\Delta\end{pmatrix}, \tag{6}$$

where

$$\epsilon_k=(\Delta^2+v_0^2k^2)^{1/2} \tag{7}$$

defines the energy spectrum of the relativistic model given by Eq. (1) with $M=0$, i.e., $E_{v\mathbf{k}}^0=-\epsilon_k$ for the valence band and $E_{c\mathbf{k}}^0=\epsilon_k$ for the conduction band—each of them being doubly degenerate. We note that the linear approximation leads to correct results when $M\ll\Delta$ and the Fermi level is not too close to the band edges; $\mu-\Delta\geq M$ for μ in the conduction band and $\mu-\Delta\leq-M$ for μ in the valence band. When these conditions are not obeyed, the linear approximation may not be sufficient to describe properly the Hall conductivity.

After substituting Eq. (6) into Eq. (5) and calculating the trace, one finds

$$\sigma_{xy}^F=\sigma_{xy}^{(1)}+\sigma_{xy}^{(2)}, \tag{8}$$

where

$$\sigma_{xy}^{(1)}=\frac{iMe^2v_0^2}{\pi\hbar}(2g\mu-g^*\Delta)\times\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mu^2-\epsilon_k^2}{[(\mu+i\eta)^2-\epsilon_k^2]^2[(\mu-i\eta)^2-\epsilon_k^2]}, \tag{9}$$

$$\sigma_{xy}^{(2)}=-\frac{iMe^2v_0^2}{\pi\hbar}(2g\mu-g^*\Delta)\times\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\mu^2-\epsilon_k^2}{[(\mu+i\eta)^2-\epsilon_k^2][(\mu-i\eta)^2-\epsilon_k^2]^2}, \tag{10}$$

and we introduced the notation

$$g=(g_c+g_v)/2,\quad g^*=(g_v-g_c). \tag{11}$$

When the Fermi level is inside the upper (conduction) band, $\mu>\Delta>0$, one can calculate the integrals in Eqs. (9) and (10)

as shown in Appendix A, and finally one arrives at the following expression for the Hall conductivity:

$$\sigma_{xy}^{F} = -\frac{3Me^2(2g\mu - g^*\Delta)}{16\pi^2 v_0 \hbar(\mu^2 - \Delta^2)^{1/2}}. \tag{12}$$

In the limit of $\mu \gg \Delta$ one finds

$$\sigma_{xy}^{F} \simeq -\frac{3gMe^2}{8\pi^2 v_0 \hbar}. \tag{13}$$

When the Fermi level is close to the conduction-band edge, $\mu = \Delta + \delta$ and $0 < \delta \ll \Delta$, we obtain from Eq. (12)

$$\sigma_{xy}^{F} \simeq -\frac{3Me^2(2g - g^*)m^{1/2}}{16\pi^2\hbar^2(2\delta)^{1/2}}, \tag{14}$$

where m is the effective mass at the band edge, $m = \hbar^2\Delta/v_0^2$. Above, δ can be considered as the chemical potential measured from the bottom of the conduction band. However, we recall that result (14) may not describe properly the Hall conductivity when the Fermi level is very close to the conduction-band edge, i.e., $\delta < M$.

Result (13) resembles that found for the nonrelativistic case within the side-jump mechanism[2,25]

$$\sigma_{xy}^{(\mathrm{sj})} = \frac{e^2}{6\hbar}\lambda_0^2(\nu_\downarrow \hbar k_{F\downarrow} v_{F\downarrow} - \nu_\uparrow \hbar k_{F\uparrow} v_{F\uparrow}), \tag{15}$$

where λ_0 is the spin-orbit coupling parameter, $\nu_{\uparrow,\downarrow} = mk_{F\uparrow,\downarrow}/(2\pi^2\hbar^2)$ are the densities of states in the spin-up and spin-down subbands, $k_{F\uparrow,\downarrow}$ are the corresponding Fermi wave vectors, $v_{F\uparrow,\downarrow}$ are the Fermi velocities, and m_0 is the free electron mass. If we take $v_{F\uparrow} = v_{F\downarrow} = v_F$ and use the relation $g_c M = \hbar^2(k_{F\uparrow}^2 - k_{F\downarrow}^2)/(2m)$, we find

$$\sigma_{xy}^{(\mathrm{sj})} = -\frac{g_c M e^2}{6\pi^2\hbar^4} m^2 v_F \lambda_0^2. \tag{16}$$

Comparing Eqs. (13) and (16) we see that the result for nonrelativistic case (16) approaches that in Eq. (13) for $\lambda_0^2 \simeq 9\hbar^3/(4m^2 v_F v_0)$.

When the Fermi level is in the valence band, the Hall conductivity can be found in a similar way and is still given by Eq. (12). In the limit of $\mu \ll -\Delta$ one then finds

$$\sigma_{xy}^{F} \simeq \frac{3gMe^2}{8\pi^2 v_0 \hbar}. \tag{17}$$

When in turn the Fermi level is close to the top of the valence band, $\mu = -\Delta - \delta'$ and $0 < \delta' \leq \Delta$, one finds from Eq. (12)

$$\sigma_{xy}^{F} \simeq \frac{3Me^2(2g + g^*)m^{1/2}}{16\pi^2\hbar^2(2\delta')^{1/2}}, \tag{18}$$

where δ' can be considered as the chemical potential measured from the top of the valence band toward negative-energy values. As before, result (18) may not describe properly the conductivity when $\delta' < M$.

When magnetic splitting in the conduction band is the same as in the valence band, i.e., $g_c = g_v = g$ or $g^* = 0$, the symmetry between valence and conduction bands of the Dirac model is restored. If additionally the Fermi level is measured from the middle of the energy gap, this particle-hole symmetry leads to the relation $\sigma_{xy}^{F}(\mu) = -\sigma_{xy}^{F}(-\mu)$. In other words, when the Fermi level is in the conduction band, the corresponding Hall conductivity is equal to minus the Hall conductivity for the Fermi level lying in the valence band symmetrically with respect to the middle of the gap. The formulas presented above for σ_{xy}^{F} in some specific situations are in agreement with this general relation.

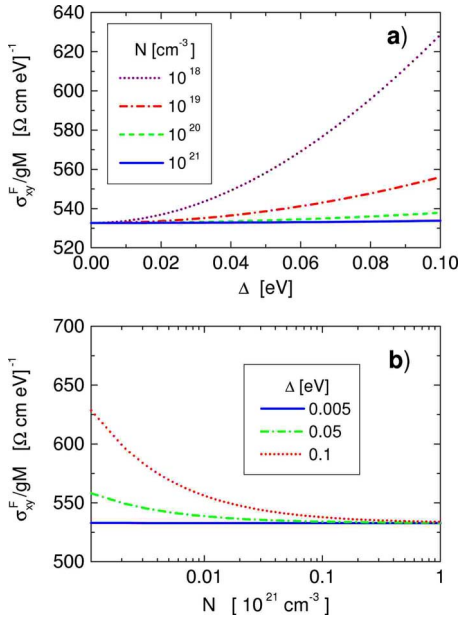

FIG. 1. (Color online) Hall conductivity from states at the Fermi surface in the case when the Fermi level is located in the valence band, calculated as a function of the (a) parameter Δ and (b) impurity concentration N. The other parameters are $v_0 = 5 \times 10^{-8}$ eV cm and $g_c = g_v = g$.

In Fig. 1 we show the Hall conductivity σ_{xy}^{F} as a function of the energy gap (parameter Δ) and impurity (hole) concentration N. For simplicity, it has been assumed that each impurity creates one hole (electron) for the Fermi level inside the valence (conduction) band. We note that the impurity concentration determines the Fermi level (in the valence band for the case shown in Fig. 1) at which the conductivity is calculated, while the associated impurities do not play any essential role, i.e., the contribution due to impurity scattering is assumed to be small and therefore neglected. The absolute magnitude of the Hall conductivity increases with increasing parameter Δ and decreasing N. This can be accounted for by noting that the density of states increases when the energy tends to the band edge and also increases with increasing Δ.

III. TOPOLOGICAL CONTRIBUTION TO AHE

Now, we calculate the topological contribution within the Kubo formalism.[15] This leads to the same result as the calculation of Berry phase in the momentum space. Let us begin with the general formula for the frequency-dependent conductivity $(\omega \neq 0)$,[26]

$$\sigma_{xy}(\omega) = \frac{e^2\hbar}{\omega}\mathrm{Tr}\int \frac{d\varepsilon}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3} v_x G_{\mathbf{k}}(\varepsilon+\omega) v_y G_{\mathbf{k}}(\varepsilon), \quad (19)$$

where v_x and v_y are defined in Eq. (2).

We have calculated Eq. (19) in the limit of small magnetization M (for details see Appendix B), and the result can be written as

$$\sigma_{xy} = \sigma_{xy}^{v} + \sigma_{xy}^{c}, \quad (20)$$

where σ_{xy}^{v} and σ_{xy}^{c} are contributions from the valence and conduction bands,

$$\sigma_{xy}^{\alpha} = -\frac{2e^2v_0^2gM}{3\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3} f(E_{\alpha\mathbf{k}}^0)\left[\frac{P_0'''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^4} - \frac{12P_0''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^5} + \frac{60P_0'(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^6} - \frac{120P_0(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^7}\right] \quad (21)$$

for $\alpha = v, c$. Here $f(\varepsilon)$ is the Fermi distribution (for $T=0$ in our case), $P_0(\varepsilon)$ is defined as $P_0(\varepsilon) = dP(\varepsilon_1,\varepsilon_2)/d\varepsilon_2|_{\varepsilon_1,\varepsilon_2=\varepsilon}$, where $P(\varepsilon_1,\varepsilon_2)$ is defined by Eq. (B2) in Appendix B, and $P_0'(\varepsilon)$ to $P_0''''(\varepsilon)$ is the first to fourth derivative of $P_0(\varepsilon)$, respectively.

Symmetry of the Dirac model assures that the integral in the above equation from the conduction-band bottom to the Fermi level μ in the conduction band [assuming $f(\varepsilon)=1$] is equal to minus the integral from the valence-band top to the Fermi level $-\mu$ in the valence band. This allows us to write the total Hall conductivity as $\sigma_{xy} = \sigma_{xy}^0 - \delta\sigma_{xy}$ when the Fermi level is in the valence band. Here $\delta\sigma_{xy}$ is the integral from the Fermi level to the top of the valence band, which has to be subtracted from the contribution σ_{xy}^0 of the fully occupied valence band. In turn, when the Fermi level is inside the conduction band, one may write $\sigma_{xy} = \sigma_{xy}^0 + \delta\sigma_{xy}$, where $\delta\sigma_{xy}$ is now the contribution from the occupied part of the conduction band. However, as mentioned above, $\delta\sigma_{xy}(\mu>0) = -\delta\sigma_{xy}(-\mu)$, which shows that the conductivity is invariant with respect to the sign change in the Fermi level. Thus, the following numerical results will be presented for the case when the Fermi level is in the valence band, and we write the conductivity as

$$\sigma_{xy} = \sigma_{xy}^0 - \Delta\sigma_{xy}. \quad (22)$$

The term σ_{xy}^0 is independent of the Fermi-level position (the contribution from the fully occupied valence band) while $\Delta\sigma_{xy}$ is Fermi-level dependent. We note that nonvanishing topological contributions to the spin and anomalous Hall effects have already been discussed in the relevant literature (see, for example, Ref. 27 and references therein).

In Fig. 2 we show variation in $\Delta\sigma_{xy}$ with the parameter Δ as well as with the impurity (hole) density N. As one can see, $\Delta\sigma_{xy}$ is positive and increases with increasing hole concentration, which is rather clear taking into account the fact that by definition $\Delta\sigma_{xy}=0$ for $N=0$. However, $\Delta\sigma_{xy}$ decreases with increasing parameter Δ (energy gap).

In Fig. 3(b) we show density plots for $\Delta\sigma_{xy}$ calculated as a function of the energy gap (parameter Δ) and hole concentration N. For comparison we also show there in Fig. 3(a) the corresponding density plot for the Hall conductivity σ_{xy}^F at the Fermi level in the valence band (see Sec. II). We define

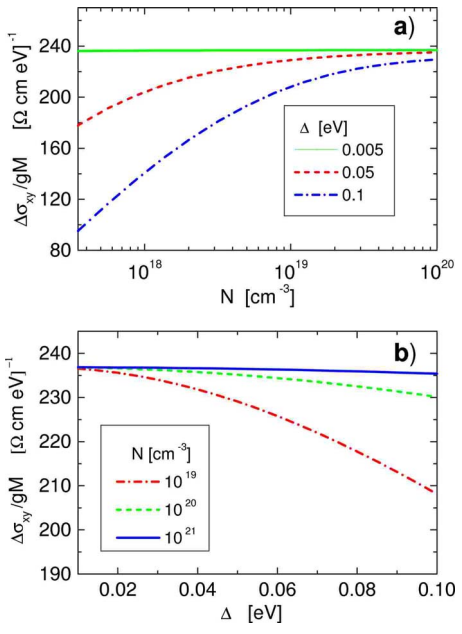

FIG. 2. (Color online) The term $\Delta\sigma_{xy}$ calculated as a function of (a) the parameter Δ and (b) the impurity (hole) concentration N. The other parameters are as in Fig. 1.

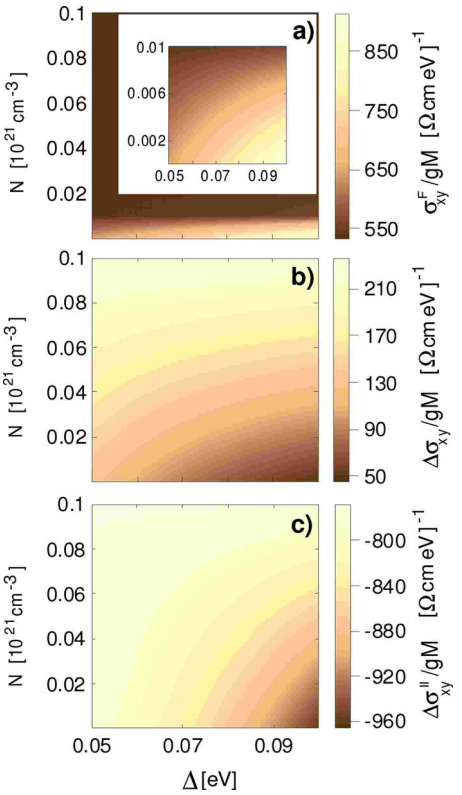

FIG. 3. (Color online) (a) Hall conductivity at the Fermi level in the valence band, (b) the term $\Delta\sigma_{xy}$, and (c) the term $\Delta\sigma_{xy}^{\mathrm{II}} = -\Delta\sigma_{xy} - \sigma_{xy}^F$. All quantities are calculated as a function of the energy gap (parameter Δ) and the impurity (hole) concentration N. The other parameters are as in Fig. 1.

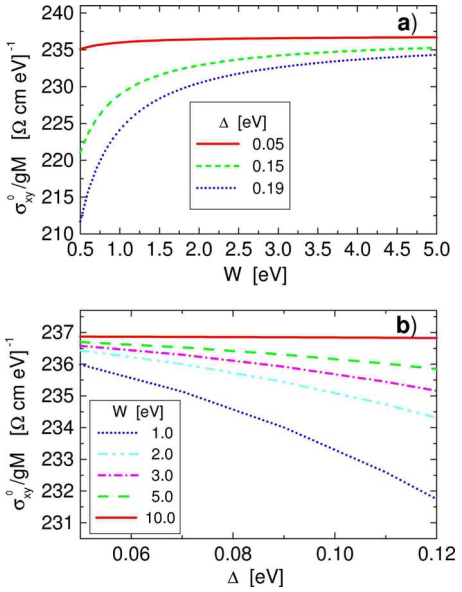

FIG. 4. (Color online) Contribution σ_{xy}^0 to the Hall conductivity calculated (a) as a function of the band width W for indicated values of the parameter Δ and (b) as a function of the parameter Δ for indicated values of the valence-band width W. The other parameters are the same as in Fig. 1.

$\sigma_{xy}^{\mathrm{II}}$ as $\sigma_{xy}^{\mathrm{II}}=\sigma_{xy}-\sigma_{xy}^F$. Thus, $\sigma_{xy}^{\mathrm{II}}$ may be written as $\sigma_{xy}^{\mathrm{II}}=\sigma_{xy}^0+\Delta\sigma_{xy}^{\mathrm{II}}$, where $\Delta\sigma_{xy}^{\mathrm{II}}$ is defined as $\Delta\sigma_{xy}^{\mathrm{II}}=-\Delta\sigma_{xy}-\sigma_{xy}^F$. The term $\Delta\sigma_{xy}^{\mathrm{II}}$ is shown in Fig. 3(c).

Since the valence band in the Dirac model assumed here is not bounded from the bottom, we have to impose some cutoff band edge to calculate σ_{xy}^0. The corresponding contribution from the whole valence band depends on the assumed band width W and also on the parameter Δ, as shown in Fig. 4. We note that the constant term of the off-diagonal conductivity, σ_{xy}^0, is due to the topology of the fully occupied valence band. Therefore, if calculated properly, it does not depend on any numerical parameters. As we see from Fig. 4, σ_{xy}^0 is independent of Δ for $W=10$ eV. In Fig. 5 we show density plots for σ_{xy}^0 as a function of both Δ and W.

IV. GEOMETRIC INTERPRETATION

In this section we discuss the topology of electron energy bands of the Dirac model and its relation to the intrinsic mechanism of AHE. It is known[8,9] that the topological contribution can be presented in the form of an integral over $\mathbf{k}$ space of the Berry curvature[28,29] $\mathbf{\Omega}(m\mathbf{k})$ of the mth electron energy band defined as

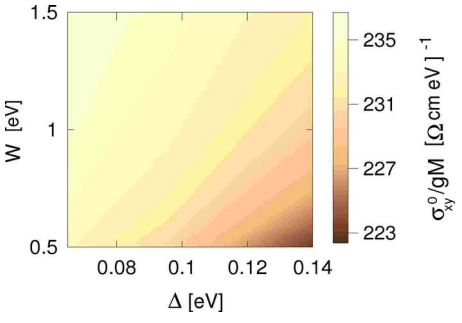

FIG. 5. (Color online) Contribution σ_{xy}^0 to the Hall conductivity calculated as a function of the parameter Δ and band width W. The other parameters are the same as in Fig. 1.

$$\Omega^i(m\mathbf{k})=\epsilon^{ijl}\frac{\partial}{\partial k_j}\mathcal{A}^l(m\mathbf{k}), \tag{23}$$

where $\mathcal{A}^i(m\mathbf{k})$ is the so-called *connection*,

$$\mathcal{A}^i(m\mathbf{k})=-i\langle m\mathbf{k}|\frac{\partial}{\partial k_i}|m\mathbf{k}\rangle, \tag{24}$$

and $|m\mathbf{k}\rangle$ is an eigenvector of the Hamiltonian H.

If the magnetization is absent, $M=0$, the eigenvectors of Dirac Hamiltonian can be easily found as

$$\langle 1\mathbf{k}|=N_v[-v_0k_-,v_0k_z,0,(\epsilon_k+\Delta)],$$

$$\langle 2\mathbf{k}|=N_v[-v_0k_z,-v_0k_+,(\epsilon_k+\Delta),0], \tag{25}$$

$$\langle 3\mathbf{k}|=N_c[v_0k_-,-v_0k_z,0,(\epsilon_k-\Delta)],$$

$$\langle 4\mathbf{k}|=N_c[v_0k_z,v_0k_+,(\epsilon_k-\Delta),0], \tag{26}$$

where $N_{v,c}=[2\epsilon_k(\epsilon_k\pm\Delta)]^{-1/2}$ are the normalization factors and $k_\pm=k_x\pm ik_y$. The wave functions $|1\mathbf{k}\rangle$ and $|2\mathbf{k}\rangle$ correspond to the energy states in the valence band with energy $\varepsilon=E_{v\mathbf{k}}^0=-\epsilon_k$, whereas $\langle 3\mathbf{k}|$ and $\langle 4\mathbf{k}|$ to the states in the conduction band of energy $\varepsilon=E_{c\mathbf{k}}^0=\epsilon_k$. Using the above wave functions and Eqs. (23) and (24) we find that the Berry curvature for the states in the two valence subbands is nonzero. However, the corresponding contributions to σ_{xy} from different subbands compensate each other in the nonmagnetic case. The same also applies to the conduction subbands. But if we include the nonzero magnetization M, which induces a band splitting, one can expect a nonzero resulting effect.

To make it clearer, let us consider a general case of the Hamiltonian

$$H(\mathbf{n})=\begin{pmatrix}\Delta-g_cM\sigma_z & \lambda\mathbf{n}\cdot\boldsymbol{\sigma}\\ \lambda\mathbf{n}\cdot\boldsymbol{\sigma} & -\Delta-g_vM\sigma_z\end{pmatrix}, \tag{27}$$

where $\mathbf{n}$ is a unit vector that depends on $\mathbf{k}$ and $\lambda(k)$ is a scalar. In the case of Dirac Hamiltonian (1), the vector $\mathbf{n}=\mathbf{k}/k$ coincides with a unit vector along the wave vector $\mathbf{k}$ and $\lambda=v_0k$. The presentation of Hamiltonian in form (27) means a parametrization of the Hamiltonian with the vector field $\mathbf{n}(\mathbf{k})$.[30] Correspondingly, the eigenvectors of $H(\mathbf{n})$ are also parametrized with $\mathbf{n}$, so that the Hilbert space of the eigenvectors of H is mapped on the Berry sphere, and the points on this sphere correspond to electron states.

At each point of the space $\mathbf{k}$, the vector $\mathbf{n}$ is oriented differently, and one can use local gauge transformation $T(\mathbf{k})$ to make the $\mathbf{n}$ field homogeneous and oriented along the axis z. Such a transformation $T(\mathbf{k})$ is defined by

$$T^{-1}(\mathbf{k})H(\mathbf{n})T(\mathbf{k})=H(\mathbf{n}_0), \tag{28}$$

where $\mathbf{n}_0$ is a unit vector along the axis z, $\mathbf{n}_0=(0,0,1)$. The eigenfunctions of $H(\mathbf{n})$ are related with those of $H(\mathbf{n}_0)$, which we denote by $|m0\rangle$ as they do not depend on $\mathbf{k}$. Thus, we get $|m\mathbf{k}\rangle=T(\mathbf{k})|m0\rangle$. Hence, this approach just relates the

gauge field $\mathcal{A}^i(m\mathbf{k})$ to the corresponding gauge transformation,

$$\mathcal{A}^i(m\mathbf{k}) = -i\langle m0|T^{-1}(\mathbf{k})\frac{\partial}{\partial k_i}T(\mathbf{k})|m0\rangle, \tag{29}$$

and the Berry curvature is the gauge field analogous to the electromagnetic field in quantum electrodynamics.[31]

The above gives us another way to study the topological contribution to AHE; instead of calculating connection (24) and curvature (23) using the eigenfunctions, one can consider topological properties of the mapping determined by the dependence $\mathbf{n}(\mathbf{k})$. In some cases, e.g., for completely filled energy bands in the 2D case or for the quantum Hall effect, this gives the condition of quantization related to the topological charge.[15,30,32]

Using Hamiltonian (27) we calculated the off-diagonal conductivity in the approximation linear in M (for details see Appendix C),

$$\sigma_{xy} = \frac{4ie^2gM}{\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d\varepsilon}{2\pi} \times \frac{(\Lambda_{xy}-\Lambda_{yx})\mathcal{A}(\varepsilon+\mu)+\lambda^2 n_z\epsilon_{\gamma\mu\nu}n_\gamma\frac{\partial n_\mu}{\partial k_x}\frac{\partial n_\nu}{\partial k_y}\mathcal{B}(\varepsilon+\mu)}{[\varepsilon-\epsilon_k+\mu+i\eta\,\mathrm{sgn}\,\varepsilon]^4[\varepsilon+\epsilon_k+\mu+i\eta\,\mathrm{sgn}\,\varepsilon]^4}. \tag{30}$$

After integrating over ε we obtain the conductivity in form (20) with the contribution from band α ($\alpha=v,c$) given by

$$\begin{aligned}\sigma_{xy}^{\alpha} = &-\frac{2e^2gM}{3\hbar}\int\frac{d^3\mathbf{k}}{(2\pi)^3}f(E_{\alpha\mathbf{k}}^0)(\Lambda_{xy}-\Lambda_{yx})\left[\frac{\mathcal{A}'''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^4}\right.\\ &\left.-\frac{12\mathcal{A}''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^5}+\frac{60\mathcal{A}'(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^6}-\frac{120\mathcal{A}(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^7}\right]\\ &-\frac{2e^2gM}{3\hbar}\int\frac{d^3\mathbf{k}}{(2\pi)^3}f(E_{\alpha\mathbf{k}}^0)\lambda^2 n_z\epsilon_{\gamma\mu\nu}n_\gamma\frac{\partial n_\mu}{\partial k_x}\frac{\partial n_\nu}{\partial k_y}\\ &\times\left[\frac{\mathcal{B}'''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^4}-\frac{12\mathcal{B}''(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^5}+\frac{60\mathcal{B}'(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^6}\right.\\ &\left.-\frac{120\mathcal{B}(E_{\alpha\mathbf{k}}^0)}{(2E_{\alpha\mathbf{k}}^0)^7}\right].\end{aligned} \tag{31}$$

Here,

$$\begin{aligned}\Lambda_{xy}-\Lambda_{yx} = \lambda\left[n_x\left(\frac{\partial\lambda}{\partial k_x}\frac{\partial n_y}{\partial k_y}-\frac{\partial n_y}{\partial k_x}\frac{\partial\lambda}{\partial k_y}\right)+n_y\left(\frac{\partial n_x}{\partial k_x}\frac{\partial\lambda}{\partial k_y}\right.\right.\\ \left.\left.-\frac{\partial\lambda}{\partial k_x}\frac{\partial n_x}{\partial k_y}\right)\right]+\lambda^2\left(\frac{\partial n_x}{\partial k_x}\frac{\partial n_y}{\partial k_y}-\frac{\partial n_y}{\partial k_x}\frac{\partial n_x}{\partial k_y}\right).\end{aligned} \tag{32}$$

The integration in k_x,k_y plane in Eq. (31) can be presented by using the mapping of this plane to the Berry sphere of $\mathbf{n}(\mathbf{k})$ with k_z=const. The corresponding spherical angles $\theta(\mathbf{k})$ and $\phi(\mathbf{k})$ defining the orientation of the vector $\mathbf{n}(\mathbf{k})$ can be used as new variables. From Eq. (32) we find

$$\begin{aligned}[\Lambda_{xy}-\Lambda_{yx}] = &\frac{\lambda\lambda'(k)\sin^2\theta}{k}\left(k_x\frac{\partial\phi}{\partial k_y}-k_y\frac{\partial\phi}{\partial k_x}\right)\\ &+\lambda^2\sin\theta\cos\theta\left(\frac{\partial\theta}{\partial k_x}\frac{\partial\phi}{\partial k_y}-\frac{\partial\theta}{\partial k_y}\frac{\partial\phi}{\partial k_x}\right).\end{aligned} \tag{33}$$

The first term in Eq. (33) contains the winding number of the angle $\phi(\mathbf{k})$ along any closed contour in the k_x,k_y plane, whereas the second term contains the Jacobian of transition from k_x,k_y to the variables θ and ϕ.

The same Jacobian also appears in the second term of Eq. (31),

$$n_z\epsilon_{ijl}n_i\frac{\partial n_j}{\partial k_x}\frac{\partial n_l}{\partial k_y} = \sin\theta\cos\theta\left(\frac{\partial\theta}{\partial k_x}\frac{\partial\phi}{\partial k_y}-\frac{\partial\theta}{\partial k_y}\frac{\partial\phi}{\partial k_x}\right), \tag{34}$$

which allows to present the integration over k_x and k_y in Eq. (31) as an integration over the surface of the unit sphere. The result of such integration depends on the mapping $n(\mathbf{k})$ and on all other functions of $\mathbf{k}$ in Eq. (31), which should be written now as functions of new variables k_z, θ, and ϕ.

Turning back to the Dirac model with $\mathbf{n}(\mathbf{k})=\mathbf{k}/k$, we find that the first term in Eq. (33) contains the winding number +1 along any contour around $k_x=k_y=0$, which gives 2π for integral over angles in the k_x,k_y plane. Calculating Eq. (33) we find that it is equal to +1, and Eq. (34) gives us $k_z^2/k^2=\cos^2\theta$. This corresponds to the integration over the Berry surface, or equivalently, to the integration over k_x and k_y.

V. DISCUSSIONS

We have calculated the off-diagonal (Hall) conductivity using the relativistic Dirac model of electron energy spectrum of IV-VI magnetic semiconductors. The considerations include the contribution from the Fermi level calculated separately in Sec. II as well as the topological contribution related to the states below the Fermi energy. We found that the Hall conductivity depends on the location of Fermi level, which can be also interpreted as the dependence on electron or hole density. This dependence should be especially important in the case of magnetic semiconductors, as it makes the anomalous Hall effect somewhat similar to the ordinary Hall effect which depends strongly on the concentration of electrons or holes, and therefore is used for the characterization of semiconductors.

As follows from our calculations, the off-diagonal conductivity does not vanish when the Fermi level is located within the energy gap. The corresponding Hall conductivity, σ_{xy}^0, does not depend on any band parameters but on the magnitude of spin splitting in the valence band. In view of the strong spin-orbit interaction, it is not possible to separate contributions from the spin up and down states. As a result, both magnetically split bands contribute to σ_{xy}^0.

The analysis of topology of the electron energy bands in the Dirac model shows that the nonzero Hall conductivity of the filled valence band is related to the "hedgehoglike" topological excitation of the **n** field describing the evolution of the eigenfunctions in the inverse space. It is quite similar to the case of AHE in two-dimensional Dirac model, but in the 3D case the integration over **k** space does not present any topological charge leading to the quantization of the Hall conductivity.

In Secs. II and III we calculated the AHE in the approximation linear in M, assuming that the magnitude of magnetization in magnetic semiconductors is usually rather small. The linearity of the dependence of $\sigma_{xy}(M)$ in Mn_5Ge_3 ferromagnets has been discussed recently and interpreted within the spin-wave mechanism of AHE.[33] It should be noted that the mechanism of Ref. 33 can be essential at finite temperatures and becomes irrelevant for $T\rightarrow 0$, where the density of magnons exponentially decreases.

Our calculations do not include impurities. It is well known that impurities can affect the part of AHE related to the Fermi level but they do not influence the topological contribution. Recent *ab initio* calculations demonstrated that the Berry-phase mechanism of AHE in $CuCr_2Se_{4-x}Br_x$ is strongly prevailing even when the impurity scattering rate is changed by several orders of magnitude.[34] Therefore we think that our findings concerning the AHE in the insulating regime (with the Fermi level in the gap) should not depend on impurities. Besides, we can also expect the leading role of the topological mechanism of AHE in the case of doped semiconductors due to the large SO interaction in the crystal potential, whereas the impurity potential can be rather small.

Hamiltonian (1) describes isotropic energy spectrum near the L points of the Brillouin zone in IV-VI semiconductors. In systems under considerations there are four nonequivalent L points, and contributions from all these points have to be taken into account (for isotropic L points this leads to an additional factor of 4).

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APPENDIX A: CALCULATION OF INTEGRALS

When the Fermi level is inside the upper (conduction) band, $\mu>\Delta>0$, one can write the terms $\sigma_{xy}^{(1)}$ and $\sigma_{xy}^{(2)}$ [Eqs. (9) and (10)] in the forms

$$\sigma_{xy}^{(1)}=-\frac{iMe^2}{2\pi^3 v_0\hbar}(2g\mu-g^*\Delta)\times\int_0^\infty\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)^2(x+a+i\eta)^2(x-a+i\eta)(x+a-i\eta)} \tag{A1}$$

and

$$\sigma_{xy}^{(2)}=\frac{iMe^2}{2\pi^3 v_0\hbar}(2g\mu-g^*\Delta)\times\int_0^\infty\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)(x+a+i\eta)(x-a+i\eta)^2(x+a-i\eta)^2}, \tag{A2}$$

where $a=\sqrt{\mu^2-\Delta^2}$.

To calculate the integrals we rewrite the first one as

$$I_1=\int_0^\infty\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)^2(x+a+i\eta)^2(x-a+i\eta)(x+a-i\eta)}=\int_{\Gamma_1}\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)^2(x-a+i\eta)(x+a)^3}-\int_{-i\infty}^0\frac{x^2(a^2-x^2)dx}{(x-a)^3(x+a)^3}, \tag{A3}$$

where the closed contour Γ_1 in the complex x plane goes from $x=0$ to ∞, then in the lower complex plane to $-i\infty$, and then back to $x=0$. The second integral in Eq. (A3) compensates the part along the imaginary axis. The first integral is calculated using the simple pole at $x=a-i\eta$, and in the second one we substitute $x=-iy$. Then one finds

$$I_1=\frac{i\pi}{16a}\frac{2ia\eta+\eta^2}{\eta^2}-i\int_0^\infty\frac{y^2dy}{(y^2+a^2)^2}. \tag{A4}$$

Analogously, we present

$$I_2=\int_0^\infty\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)(x+a+i\eta)(x-a+i\eta)^2(x+a-i\eta)^2}=\int_{\Gamma_2}\frac{x^2(a^2-x^2)dx}{(x-a-i\eta)(x-a+i\eta)^2(x+a)^3}+\int_0^{i\infty}\frac{x^2(a^2-x^2)dx}{(x-a)^3(x+a)^3}, \tag{A5}$$

where the contour Γ_2 goes from $x=0$ to ∞, then in the upper half plane to $i\infty$, and then back to $x=0$. For I_2 we obtain

$$I_2=-\frac{i\pi}{16a}\frac{-2ia\eta+\eta^2}{\eta^2}+i\int_0^\infty\frac{y^2dy}{(y^2+a^2)^2}. \tag{A6}$$

Thus,

$$I_1-I_2=\frac{i\pi}{8a}-2i\int\frac{y^2dy}{(y^2+a^2)^2}=-\frac{3i\pi}{8a}. \tag{A7}$$

This leads to the final expression (12).

APPENDIX B: CALCULATION OF THE TOPOLOGICAL TERM

Using the Green's functions of Hamiltonian (1) and assuming $g_c=g_v=g$, one finds from Eq. (19)

$$\sigma_{xy}(\omega)=\frac{4ie^2v_0^2gM}{\omega\hbar}\int\frac{d\varepsilon}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3}P(\varepsilon+\mu+\omega,\varepsilon+\mu)\times\frac{1}{\prod_{m=1}^{4}[\varepsilon+\omega-E_{m\mathbf{k}}+\mu+i\eta\,\mathrm{sgn}(\varepsilon+\omega)]}\times\frac{1}{\prod_{n=1}^{4}[\varepsilon-E_{n\mathbf{k}}+\mu+i\eta\,\mathrm{sgn}(\varepsilon)]},\tag{B1}$$

where

$$\begin{aligned}P(\varepsilon_1,\varepsilon_2)=(\varepsilon_2-\varepsilon_1)[&-3\Delta^4+\varepsilon_1^2(\varepsilon_2^2-v_0^2k_t^2+v_0^2k_z^2-g^2M^2)\\&+\Delta^2(\varepsilon_1^2+\varepsilon_2^2-2v_0^2k_t^2-6v_0^2k_z^2+2g^2M^2)+v_0^4k_t^4\\&-3v_0^4k_z^4+2v_0^2k_z^2g^2M^2+g^4M^4-2v_0^2k_t^2(v_0^2k_z^2\\&-g^2M^2)-\varepsilon_2^2(v_0^2k_t^2-v_0^2k_z^2+g^2M^2)].\end{aligned}\tag{B2}$$

Here, $k_t=(k_x^2+k_y^2)^{1/2}$ is the perpendicular component of the wave vector. The energy spectrum consists of the following four bands:

$$E_{1,4\mathbf{k}}=\mp[\Delta^2+v_0^2k^2+g^2M^2+2gM(\Delta^2+v_0^2k_z^2)^{1/2}]^{1/2},\tag{B3}$$

$$E_{2,3\mathbf{k}}=\mp[\Delta^2+v_0^2k^2+g^2M^2-2gM(\Delta^2+v_0^2k_z^2)^{1/2}]^{1/2}.\tag{B4}$$

When $M=0$, the bands $E_{1,2\mathbf{k}}$ reduce to $E^0_{v\mathbf{k}}$, while the bands $E_{3,4\mathbf{k}}$ reduce to $E^0_{c\mathbf{k}}$.

Taking the limit of $\omega\to0$ and considering the term linear in M, one arrives at the following formula for the Hall conductivity:

$$\sigma_{xy}=\frac{4ie^2v_0^2gM}{\hbar}\int\frac{d\varepsilon}{2\pi}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{P_0(\varepsilon+\mu)}{[\varepsilon-\epsilon_k+\mu+i\eta\,\mathrm{sgn}\,\varepsilon]^4}\times\frac{1}{[\varepsilon+\epsilon_k+\mu+i\eta\,\mathrm{sgn}\,\varepsilon]^4},\tag{B5}$$

where $P_0(\varepsilon)=dP(\varepsilon_1,\varepsilon_2)/d\varepsilon_2|_{\varepsilon_1,\varepsilon_2=\varepsilon}$. After integrating over ε one arrives at Eq. (20).

APPENDIX C: CALCULATION OF EQ. (30)

The off-diagonal conductivity for the Hamiltonian $H(\mathbf{n})$ given by Eq. (27) can be calculated in a way similar to that in Sec. III. Accordingly, we define the velocity operator

$$v_i=\frac{1}{\hbar}\frac{\partial H(\mathbf{n})}{\partial k_i}=\frac{1}{\hbar}\left(\frac{\partial\lambda}{\partial k_i}n_\mu+\lambda\frac{\partial n_\mu}{\partial k_i}\right)\begin{pmatrix}0&\sigma_\mu\\\sigma_\mu&0\end{pmatrix}.\tag{C1}$$

The conductivity is then calculated using Eq. (26), with the Green's function of the form

$$G(\varepsilon)=\frac{g(\varepsilon+\mu)}{\prod_{n=1}^{4}[\varepsilon-E_{n\mathbf{k}}+\mu+i\eta\,\mathrm{sgn}(\varepsilon)]},\tag{C2}$$

where the nominator $g(\varepsilon)$ of the Green's function is not shown explicitly.

As before, we assume $g_c=g_v=g$ and limit our following considerations to the contribution linear in M. At first, we calculate the trace of $v_xg(\varepsilon_1)v_yg(\varepsilon_2)$,

$$\mathrm{Tr}\{v_xg(\varepsilon_1)v_yg(\varepsilon_2)\}=\frac{1}{\hbar^2}\Lambda_{\mu\nu}\,\mathrm{Tr}\begin{pmatrix}0&\sigma_\mu\\\sigma_\mu&0\end{pmatrix}g(\varepsilon_1)\times\begin{pmatrix}0&\sigma_\nu\\\sigma_\nu&0\end{pmatrix}g(\varepsilon_2),\tag{C3}$$

where

$$\Lambda_{\mu\nu}=\left[\frac{\partial\lambda}{\partial k_x}\frac{\partial\lambda}{\partial k_y}n_\mu n_\nu+\lambda n_\mu\frac{\partial\lambda}{\partial k_x}\frac{\partial n_\nu}{\partial k_y}+\lambda n_\nu\frac{\partial n_\mu}{\partial k_x}\frac{\partial\lambda}{\partial k_y}+\lambda^2\frac{\partial n_\mu}{\partial k_x}\frac{\partial n_\nu}{\partial k_y}\right].\tag{C4}$$

Taking into account Eq. (C3) one finds

$$\mathrm{Tr}\{v_xg(\varepsilon_1)v_yg(\varepsilon_2)\}=\frac{4igM}{\hbar^2}\{(\Lambda_{xy}-\Lambda_{yx})A^{xy}+(\Lambda_{zx}-\Lambda_{xz})A^{zx}+(\Lambda_{yz}-\Lambda_{zy})A^{yz}\},\tag{C5}$$

where $A^{\mu\nu}$ are some functions of ε_1 and ε_2. Now, we expand $A^{\mu\nu}(\varepsilon_1,\varepsilon_2)$ with respect to $\omega=\varepsilon_2-\varepsilon_1$,

$$A^{\mu\nu}(\varepsilon_1,\varepsilon_2)\simeq A^{\mu\nu}|_{\varepsilon_2=\varepsilon_1=\varepsilon}+\omega\left.\frac{\partial A^{\mu\nu}}{\partial\varepsilon_2}\right|_{\varepsilon_2=\varepsilon_1=\varepsilon}.\tag{C6}$$

Calculating the coefficients of this expansion one finds

$$A^{\mu\nu}|_{\varepsilon_2=\varepsilon_1=\varepsilon}=0,\qquad\left.\frac{\partial A^{\mu\nu}}{\partial\varepsilon_2}\right|_{\varepsilon_2=\varepsilon_1=\varepsilon}=A_0^{\mu\nu}(\varepsilon),\tag{C7}$$

where

$$\begin{aligned}A_0^{xy}(\varepsilon)&=\mathcal{A}(\varepsilon)+n_z^2\mathcal{B}(\varepsilon),\\A_0^{zx}(\varepsilon)&=n_zn_y\mathcal{B}(\varepsilon),\\A_0^{yz}(\varepsilon)&=n_zn_x\mathcal{B}(\varepsilon),\end{aligned}\tag{C8}$$

and

$$\mathcal{A}(\varepsilon) = \varepsilon^4 + 2(\Delta^2 - \lambda^2)\varepsilon^2 - (2\Delta^2 - \lambda^2)\lambda^2 - 3\Delta^4, \quad \text{(C9)}$$

$$\mathcal{B}(\varepsilon) = 4\lambda^2(\varepsilon^2 - \Delta^2 - \lambda^2). \quad \text{(C10)}$$

Upon substituting Eqs. (C6)–(C10) into Eq. (C5) one arrives at the expression

$$\mathrm{Tr}\{v_x g(\varepsilon+\omega) v_y g(\varepsilon)\} = \frac{4igM\omega}{\hbar^2}(\Lambda_{xy} - \Lambda_{yx})\mathcal{A}(\varepsilon) + \frac{4igM\omega}{\hbar^2}\lambda^2 n_z \epsilon_{\gamma\mu\nu} n_\gamma \frac{\partial n_\mu}{\partial k_x}\frac{\partial n_\nu}{\partial k_y}\mathcal{B}(\varepsilon), \quad \text{(C11)}$$

which leads to result (30).

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